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STRESSES IN A TRANSVERSELY ISOTROPIC CONICAL ELASTIC PIPE OF CONSTANT THICKNESS UNDER A THERMAL LOAD†

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The method of perturbing the shape of the boundary was used to determine the stress state of thick-walled conical and biconical isotropic shells [1, 2], under the assumption that the shells are closed and that their shape deviates but little from a spherical one. In the present paper the method of successive approximations is used to obtain the solution of the problem of thermal loading of an elastic, transversely isotropic conical pipe (generally speaking, truncated) of constant thickness. The axis of symmetry of the material in question with curvilinear anisotropy is directed along the thickness of the pipe. The formulas for determining the stress state of the conical pipe at every iteration step are written in an orthogonal system of coordinates appropriate to the body in question. The first three approximations of the temperature problem are solved. The numerical results obtained confirm the good practical convergence of the method used, for a wide range of values of the geometrical parameters of a conical pipe.

1. DERIVATION OF THE BASIC EQUATIONS DESCRIBING THE DEFORMATION OF A CONICAL PIPE OF CONSTANT THICKNESS

WE SHALL understand by the term "conical pipe" a solid of revolution bounded by parallel conical surfaces, with an aperture angle of 2φ (Fig. 1). We will assume that in the pipe in questions (which is, generally, truncated), the axes of symmetry of the transversely isotropic material studied are directed along the thickness of the body.

Let us introduce the orthogonal conical system of coordinates η, θ, x (Fig. 1) natural for the pipe in question. The η and x axes are directed, respectively, along the thickness and generatrix of the pipe, and θ is the polar angle. We will place the origin of coordinates at the point O. The inner and outer side surfaces are described, respectively, by the equations $\eta = 0$ and $\eta = \eta_0$, and the cross-sections $x = x_0$ and $x = x_1$ are the ends of the pipe.

In what follows, we shall attach to the directions η, θ, x the indices 1, 2, 3.

Let us write the relation connecting the conical and cylindrical coordinates (r, θ, z)

$$r=\eta\cos\varphi+x\sin\varphi,\quad \theta=\theta,\quad z=-\eta\sin\varphi+x\cos\varphi$$

We therefore have the following relations in the $x^1 = \eta$, $x^2 = \theta$, $x^3 = x$ system of coordinates:

$$ds^2=dr^2+r^2d\theta^2+dz^2=d\eta^2+r^2d\theta^2+dx^2,$$
$$g_{11}=g_{33}=1,\quad g_{22}=r^2=(\eta\cos\varphi+x\sin\varphi)^2,\quad g_{ij}=0\quad (i\neq j)$$

where ds is the length of the elementary vector and g_{ij} is the metric tensor.

Non-zero Christoffel symbols are calculated in the orthogonal system of coordinates from the formulas

$$\Gamma^2{}_{21}=\frac{\cos\varphi}{r},\quad \Gamma_{23}{}^2=\frac{\sin\varphi}{r},\quad \Gamma_{22}{}^1=-r\cos\varphi,\quad \Gamma_{22}{}^3=-r\sin\varphi$$

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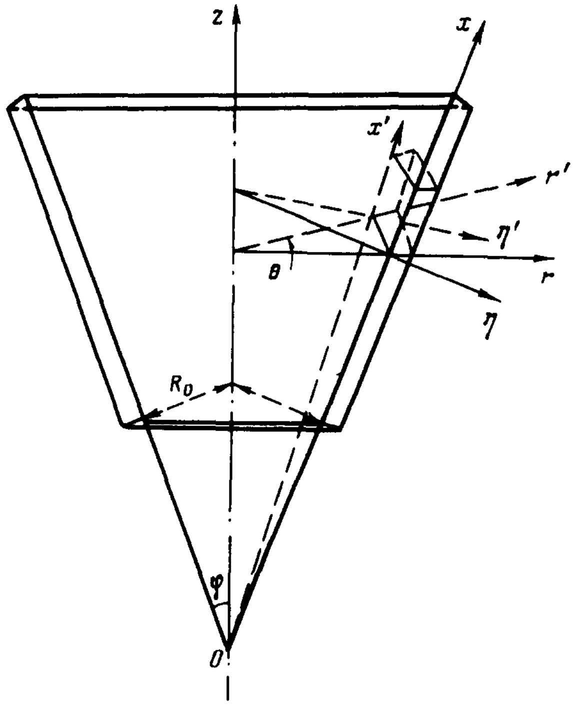

FIG. 1.

and the Christoffel symbols are symmetrical with respect to the subscripts.

The Cauchy relations for the case of axisymmetric deformation [3] take the form

$$\varepsilon_{11}=\frac{\partial u_1}{\partial \eta},\quad \varepsilon_{33}=\frac{\partial u_3}{\partial x},\quad \varepsilon_{22}=r\,(u_1\cos\varphi+u_3\sin\varphi),\quad \varepsilon_{13}=\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial \eta}$$

where ε_{11}, ε_{22}, ε_{33} and $\frac{1}{2}\varepsilon_{13}$ are the covariant components of the strain tensor and u_1, u_3 are the displacements.

The equations of equilibrium [3], in case of the axisymmetric stress state, take the form

$$\frac{\partial\sigma^{11}}{\partial\eta}+\frac{\partial\sigma^{13}}{\partial x}+\frac{\sigma^{11}\cos\varphi}{r}-\sigma^{22}r\cos\varphi+\frac{\sigma^{13}\sin\varphi}{r}=0$$

$$\frac{\partial\sigma^{13}}{\partial\eta}+\frac{\partial\sigma^{33}}{\partial x}+\frac{\sigma^{13}\cos\varphi}{r}+\frac{\sigma^{33}\sin\varphi}{r}-\sigma^{22}r\sin\varphi=0$$

where σ^{ki} are the contravariant components of the stress tensor.

The equations of equilibrium and compatibility of the deformations will be written in terms of the mixed components of the stress and strain tensors as follows:

$$\frac{\partial\sigma_{\cdot1}^{1\cdot}}{\partial\eta}+\frac{\partial\sigma_{\cdot3}^{1\cdot}}{\partial x}+\frac{\sigma_{\cdot1}^{1\cdot}-\sigma_{\cdot2}^{2\cdot}}{R}+\frac{\sigma_{\cdot3}^{1\cdot}\,\mathrm{tg}\,\varphi}{R}=0$$

$$\frac{\partial\sigma_{\cdot3}^{1\cdot}}{\partial\eta}+\frac{\sigma_{\cdot3}^{1\cdot}}{R}+\frac{\partial\sigma_{\cdot3}^{3\cdot}}{\partial x}+\frac{\sigma_{\cdot3}^{3\cdot}-\sigma_{\cdot2}^{2\cdot}}{R}\,\mathrm{tg}\,\varphi=0$$

$$R\frac{\partial\varepsilon_{\cdot2}^{2\cdot}}{\partial\eta}+\varepsilon_{\cdot2}^{2\cdot}-\varepsilon_{\cdot1}^{1\cdot}=\mathrm{tg}\,\varphi\left[\varepsilon_{\cdot3}^{1\cdot}-\frac{\partial}{\partial x}(R\varepsilon_{\cdot2}^{2\cdot})+\mathrm{tg}\,\varphi\;\;\varepsilon_{\cdot3}^{3\cdot}\right]$$

$$\frac{\partial\varepsilon_{\cdot3}^{3\cdot}}{\partial\eta}=\frac{\partial}{\partial x}\varepsilon_{\cdot3}^{1\cdot}-\frac{\partial^2}{\partial x^2}(R\varepsilon_{\cdot2}^{2\cdot})+\mathrm{tg}\,\varphi\frac{\partial}{\partial x}\varepsilon_{\cdot3}^{3\cdot}$$

$$R=r/\cos\varphi=\eta+\mathrm{tg}\,\varphi\quad x$$

$$\sigma_{\cdot1}^{1\cdot}=\sigma^{11},\quad \sigma_{\cdot3}^{1\cdot}=\sigma^{13},\quad \sigma_{\cdot3}^{3\cdot}=\sigma^{33},\quad \sigma_{\cdot2}^{2\cdot}=r^2\sigma^{22}$$

$$\varepsilon_{\cdot1}^{1\cdot}=\varepsilon_{11},\quad \varepsilon_{\cdot3}^{1\cdot}=\varepsilon_{13},\quad \varepsilon_{\cdot3}^{3\cdot}=\varepsilon_{33},\quad \varepsilon_{\cdot2}^{2\cdot}=\varepsilon_{22}/r^2$$

We note that the mixed components of the stress and strain tensors written out above in the η, θ, x system of coordinates, are also the physical components of the stresses and strains, respectively [3]. Henceforth, for convenience, we shall denote the physical components of the stresses and strains by σ_{ij} and ε_{ij}, respectively.

Let us now refer the coordinates η and x to the conical radius R_0 of the lower end of the pipe (Fig. 1). Henceforth, we shall regard η and x as the dimensionless coordinates introduced here. Thus η_0 will be the relative thickness of the lower cross-section of the pipe and $x_0 = \operatorname{ctg}\varphi$.

2. A CONICAL TRANSVERSELY ISOTROPIC PIPE UNDER THE ACTION OF AN AXIAL FORCE AND TEMPERATURE

Let the pipe in question be acted upon by an axial force Q and a temperature field $\Delta T = \Delta T(\eta)$.

The relations connecting the stresses and strains for a transversely isotropic material with curvilinear anisotropy, have the following form [4]:

$$\begin{gathered} E_2\varepsilon_{11}=k\sigma_{11}-k\nu'(\sigma_{22}+\sigma_{33})+E_2\alpha_1\Delta T \\ E_2\varepsilon_{22}=\sigma_{22}-\nu\sigma_{33}-k\nu'\sigma_{11}+E_2\alpha_2\Delta T \\ E_2\varepsilon_{33}=\sigma_{33}-\nu\sigma_{22}-k\nu'\sigma_{11}+E_2\alpha_2\Delta T \\ E_2\varepsilon_{13}=\gamma\sigma_{13}, \quad k=E_2/E_1, \quad \gamma=E_2/G \end{gathered} \tag{2.1}$$

Here it is assumed that the isotropy axis of the material is directed along the axis 1 (along the pipe thickness), E_2, E_1 are the moduli of elasticity along the axes 2 and 1, G is the shear modulus, ν and ν' are Poisson's ratios, and α_1, α_2 are the coefficients of linear expansion along the axes 1 and 2, respectively.

The boundary conditions on the side surfaces are given by

$$\sigma_{11}(0,x)=\sigma_{11}(\eta_0,x)=0, \quad \sigma_{13}(0,x)=0, \quad \sigma_{13}(\eta_0,x)=0 \tag{2.2}$$

and the boundary conditions at the ends $x = x_0$, $x = x_1$ of a long conical pipe are formulated in accordance with the Saint-Venant principle

$$\int_0^{\eta_0}\sigma_{33}R\,d\eta-\frac{Q}{2\pi(R_0\cos\varphi)^2}=\operatorname{tg}\varphi\int_0^{\eta_0}\sigma_{13}R\,d\eta \tag{2.3}$$

We rewrite the equations of equilibrium and compatibility of the strains as follows

$$\frac{\partial\sigma_{11}}{\partial\eta}+\frac{\sigma_{11}-\sigma_{22}}{R}=F_1(\eta,x), \quad F_1=-\left(\frac{\partial\sigma_{13}}{\partial x}+\frac{\operatorname{tg}\varphi}{R}\sigma_{13}\right) \tag{2.4}$$

$$\frac{\partial\sigma_{13}}{\partial\eta}+\frac{\sigma_{13}}{R}+\frac{\partial\sigma_{33}}{\partial x}+\frac{\sigma_{33}-\sigma_{22}}{R}\operatorname{tg}\varphi=0 \tag{2.5}$$

$$R\frac{\partial\varepsilon_{22}}{\partial\eta}+\varepsilon_{22}-\varepsilon_{11}=\frac{\operatorname{tg}\varphi}{E_2}F_2, \quad F_2=E_2\left[\varepsilon_{13}-\frac{\partial}{\partial x}(R\varepsilon_{22})+\operatorname{tg}\varphi\,\varepsilon_{33}\right] \tag{2.6}$$

$$\frac{\partial\varepsilon_{33}}{\partial\eta}=\frac{F_3}{E_2}, \quad F_3=E_2\left[\frac{\partial}{\partial x}\varepsilon_{13}-\frac{\partial^2}{\partial x^2}(R\varepsilon_{22})+\operatorname{tg}\varphi\frac{\partial}{\partial x}\varepsilon_{33}\right] \tag{2.7}$$

Equations (2.4)–(2.7) and conditions (2.2) and (2.3) together describe the deformation of a conical pipe under the action of the axial force Q and temperature. When relations (2.4), (2.5) and (2.3) and the first two conditions of (2.2) hold, the boundary conditions $\sigma_{13}(\eta_0, x) = 0$ is satisfied automatically.

Let us first assume that $\operatorname{tg}\varphi \ll 1$, i.e. that the cone is slim and its aperture angle is small. The terms on the right-hand sides of Eqs (2.3), (2.4), (2.6) and (2.7) are proportional to $(\operatorname{tg}\varphi)^2$ and hence are small compared with the terms on the left-hand sides of the corresponding equations. This can be easily confirmed by writing the equations of the theory of elasticity in the coordinates η and $\xi = x\,\mathrm{tg}\,\varphi$ on which the stresses should depend explicitly. We shall seek the solution of the problem using the method of successive approximations, assuming that the right-hand sides of Eqs (2.3), (2.4), (2.6) and (2.7) have been computed at the previous iteration step.

Let us investigate a thin-walled conical pipe ($\eta_0/(x\,\mathrm{tg}\,\varphi \ll 1)$). It will be correct to assume, in the case of the integral boundary condition (2.3) at the end of the conical pipe that the variation in the stresses along the x axis is small compared with the variation over the coordinate η. Therefore the right-hand sides of Eqs (2.3), (2.4), (2.6) and (2.7) are small compared with the principal terms on the left-hand sides of the corresponding equations even for large values of the parameter $\mathrm{tg}\,\varphi$. It is best to seek the solution of the problem using the iteration method mentioned above. We use, as the first approximation, the solution of problem (2.2)–(2.7) in which Eqs (2.3), (2.4), (2.6) and (2.7) do not have the right-hand sides shown above.

We will write the temperature distribution $\Delta T(\eta)$ in the form

$$\Delta T(\eta) = T_0 + T_1(\eta), \quad T_0 = \mathrm{const}$$

i.e. we separate the constant component T_0.

Let us write the formulas for determining the stress state at the nth iteration step. When integrating the equations of the problem, we have assumed that

$$\omega = \sqrt{\frac{k-(kv')^2}{1-v^2}} \neq 1$$

$$\sigma_{33}^{(n)}(\eta, x) = v\sigma_{22}^{(n)}(\eta, x) + kv'\sigma_{11}^{(n)}(\eta, x) + B(x) + \int_0^\eta F_3^{(n-1)}(\rho, x)\,d\rho - E_2\alpha_2 T_1(\eta)$$

$$\sigma_{11}^{(n)}(\eta, x) = C(x)R^{\omega-1} + D(x)R^{-\omega-1} - \Pi(x) + \Phi^{(n-1)}(\eta, x)$$

$$\sigma_{22}^{(n)}(\eta, x) = C(x)R^{\omega-1}\omega - D(x)R^{-\omega-1}\omega - \Pi(x) + \Phi_1^{(n-1)}(\eta, x)$$

$$\sigma_{13}^{(n)}(\eta, x) = -\frac{1}{R}\frac{\partial}{\partial x}\int_0^\eta \sigma_{33}^{(n)}(\rho, x)R(\rho, x)\,d\rho + \frac{\mathrm{tg}\,\varphi}{R}\int_0^\eta \sigma_{22}^{(n)}(\rho, x)\,d\rho$$

$$R = \eta + x\,\mathrm{tg}\,\varphi, \quad \Pi = \frac{P + B(x)(v - kv')}{(1-v^2)(\omega^2-1)}, \quad P = E_2(\alpha_1 - \alpha_2)T_0$$

$$\Phi^{(n-1)} = \tfrac{1}{2}(I_+ - I_- + J_+ + J_-)/\omega, \quad \Phi_1^{(n-1)} = \tfrac{1}{2}(I_+ + I_- + J_+ - J_-)$$

$$I_\pm(\eta, x) = \frac{1}{1-v^2}R^{\pm\omega-1}\int_0^\eta R^{\mp\omega}(\rho, x)\Big[\mathrm{tg}\,\varphi\,F_2^{(n-1)}(\rho, x) +$$

$$+ (v - kv')\int_0^\rho F_3^{(n-1)}(t, x)\,dt + S_T(\rho, x)\Big]d\rho$$

$$J_\pm(\eta, x) = R^{\pm\omega-1}\int_0^\eta R^{\mp\omega+1}(\rho, x)\Big[\Big(\omega \pm \frac{kv'}{1-v}\Big)F_1^{(n-1)}(\rho, x) \pm$$

$$\pm \frac{v}{1-v^2}F_3^{(n-1)}(\rho, x)\Big]d\rho$$

$$S_T(\eta, x) = E_2\Big[-(1+v)R\alpha_2\frac{d}{d\eta}T_1 - (v - kv')\alpha_2 T_1 + (\alpha_1 - \alpha_2)T_1\Big]$$

The functions $C(x)$, $D(x)$ and $B(x)$ are found at every step n of the iteration process from (2.3) and the first two conditions of (2.2). The right-hand side of Eq. (2.3) and the functions written out

TABLE 1

η/η_0	σ_{11}'	σ_{22}'	σ_{33}'	σ_{13}'	σ_{11}''	σ_{22}''	σ_{33}''	σ_{13}''
0	0	−3.447	−0.686	0	0	−1.891	−0.369	0
0.25	−4.211	−1.676	−0.344	−3.362	−1.203	−0.798	−0.187	−0.707
0.50	−5.463	0.036	−0.005	−4.368	−1.387	0.090	−0.014	−0.821
0.75	−3.990	1.697	0.331	−3.195	−0.913	0.847	0.151	−0.545
1	0	3.301	0.664	0	0	1.516	0.313	0
0	0	−3.421	−0.715	0	0	−1.838	−0.448	0
0.25	−4.193	−1.650	−0.372	−3.372	−1.177	−0.754	−0.257	−0.731
0.50	−5.458	0.060	−0.029	−4.383	−1.368	0.118	−0.051	−0.858
0.75	−4.000	1.715	0.312	−3.205	−0.909	0.862	0.148	−0.573
1	0	3.321	0.647	0	0	1.525	0.326	0
0	0	−3.420	−0.716	0	0	−1.838	−0.461	0
0.25	−4.193	−1.650	−0.372	−3.372	−1.174	−0.749	−0.265	−0.745
0.50	−5.458	0.060	−0.029	−4.383	−1.366	0.122	−0.053	−0.877
0.75	−4.000	1.715	0.312	−3.206	−0.909	0.865	0.152	−0.586
1	0	3.321	0.647	0	0	1.528	0.334	0

above with the superscript $(n-1)$ are known, since they were determined at the previous, $(n-1)$th step.

The method of successive approximations was used to solve the problem of the deformation of a conical pipe under the action of uniformly distributed temperature. In this case we have

$$Q=0, \quad T_1(\eta)\equiv 0 \text{ and } \Delta T(\eta)=T_0=\text{const}$$

Three approximations $\sigma_{ij}^{(n)}$, $n=1, 2, 3$ of the temperature problem were obtained.

Table 1 shows the results of a calculation of the stresses acting over the cross-section $x=x_0$ (at the lower end of the pipe) for $k=4$, $\nu=0.2$, $k\nu'=0.3$, $\gamma=6$. The dimensionless stresses $\sigma_{11}'=10^4\sigma_{11}/P$, $\sigma_{22}'=10^2\sigma_{22}/P$, $\sigma_{33}'=10^2\sigma_{33}/P$ and $\sigma_{13}'=10^4\sigma_{13}/P$ were calculated for $\eta_0=0.067$, $\text{tg}\varphi=0.8$, and $\sigma_{11}''=10^2\sigma_{11}/P$, $\sigma_{22}''=10\sigma_{22}/P$, $\sigma_{33}''=10\sigma_{33}/P$ and $\sigma_{13}''=10^2\sigma_{13}/P$, respectively, for $\eta_0=0.4$, $\text{tg}\varphi=0.6$. The quantity $P=E_2(\alpha_1-\alpha_2)T_0$ represents the characteristic force parameter of the problem. Table 1 contains all three approximations (they are written out in sequence corresponding to the number of the approximation).

Figure 2 shows the distribution over the x axis of the stresses $\sigma_{11}'=10^4\sigma_{11}(\frac{1}{2}\eta_0, x)/P$ (line 1), $\sigma_{22}'=10^2\sigma_{22}(0,x)/P$ (points 4), $\sigma_{33}'=10^2\sigma_{33}(0,x)/P$ (points 3), and $\sigma_{13}'=10^4\sigma_{13}(\frac{1}{2}\eta_0, x)/P$ (line 2) for $k=4$,

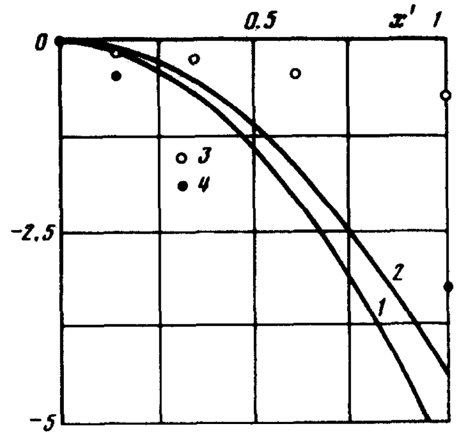

FIG. 2.

$\nu = 0.2$, $k\nu' = 0.3$, $\gamma = 6$, $\eta_0 = 0.067$, $\mathrm{tg}\,\varphi = 0.8$. The values $x' = (x\,\mathrm{tg}\,\varphi)^{-1}$ ($x_0 = \mathrm{ctg}\,\varphi$, $x \geq x_0$) are plotted along the abscissa.

The numerical results obtained show that the second approximation of the temperature problem of the deformation of a thin-walled conical pipe provides very high accuracy even at large aperture angles of the pipe φ (for example when $\mathrm{tg}\,\varphi = 0.8$). The first (asymptotic) approximation describes the stress state of a thin-walled pipe with small aperture angle with sufficient accuracy.

In the case of thick-walled pipes, the second approximation is found to provide sufficient accuracy.

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THE EFFECTIVE CHARACTERISTICS OF PIEZOACTIVE COMPOSITES WITH CYLINDRICAL INCLUSIONS†

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Using the averaging method [1, 2], a procedure is proposed for determining accurate values of the effective moduli of elasticity, piezoelectric moduli and permittivities of piezoactive composites of periodic structure with unidirectional fibres having the form of a circular cylinder. The accurate values are obtained by the analytical solution of the problems in a periodicity cell.

THE AVERAGING method has previously been used to determine the effective properties of layered piezoelectric composites in [3, 4]. To investigate the effect of the properties of fibre piezoelectric composites approximate formulas have been proposed based on a statistical approach [5] and on the method of matching and variational estimates [6].

1. Consider the non-homogeneous problem of the theory of electro-elasticity for a piezoactive composite with a periodic structure. It is described by the following system of equations [7] and boundary conditions:

$$\nabla\cdot\sigma+\mathbf{F}=0,\quad \nabla\cdot\mathbf{D}=0$$

$$\sigma=\mathbf{C}\left(\frac{x}{\varepsilon}\right)\cdot\cdot\nabla\mathbf{u}+\mathbf{e}^T\left(\frac{x}{\varepsilon}\right)\cdot\nabla\varphi \tag{1.1}$$

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